IT'S TRUE, IT'S TRUE

Clock Arithmetic: 8 and 5 Equal 1

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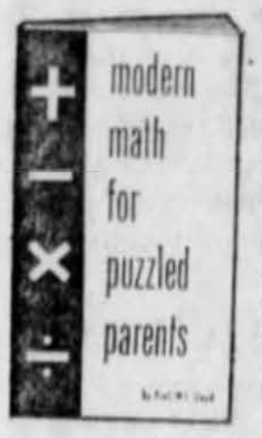
Fourth graders studying the new math bring home some arithmetic problems that cause consternation among parents not in the know. These deal with a totally different kind of numbers. With them we cannot say that 9 is larger than 2; we cannot say that 9 in the morning is later than 2 in the afternoon.



Parents will see statements like $8 + 5 = 1$, $3 - 9 = 6$, $8 \times 5 = 4$, or division problems that do not have a unique answer as in the case of $3 \div 3$, where the answer may be 1 or 5 or 9.

This is modular arithmetic, more commonly known as clock arithmetic. The numbers are modular numbers or clock numbers. The problem $8 + 5 = 1$ is not at all strange when you translate it to mean that a worker reporting at 8 o'clock and working 5 hours will knock off at 1 o'clock.

We can say loosely that modular numbers are the numbers the hour hand points to as it rotates around the dial. Thus our numbers go round and round, and we count 12, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 1, 2, 3, . . . Notice when we get to 12 we start over with 1 and do not go on to 13.

While there is no number larger than 12 on a clock face, we can relate any integer, however large, to a clock number.

Ninth of a Series

Remember we define clock numbers as the number the hour hand points to. At 14 hours after midnight the clock hand will point to 2. It will also point to 2 at 26 hours after midnight. There must be some relation between 2, 14, and 26.

WE DEFINE THIS relationship as congruence, and say that 2 is congruent to 14 and 2 is congruent to 26. This is a unique feature of modular numbers. They are not really single numbers but sets of numbers. Every number on the dial is congruent to itself plus 12, or to itself plus any multiple of 12. (5 is congruent to 17, 29, 41, 53, . . .).

Congruence is shown by a sign similar to the equal sign but having three parellel lines instead of two. The typewriter (and most newspaper type fonts) do not contain this symbol so I will use the equal sign (=) to mean congruent.

For convenience let us work with modulo 5. This calls for us to visualize a new clock face. Imagine a clock face with 0 at the top, in the place traditionally occupied by 12, and the numbers 1, 2, 3, and 4 equally spaced around the dial. (The hour hand makes a complete revolution in 5 hours). Thus we will count 0, 1, 2, 3, 4, 0, 1, 2, 3, 4, 0, 1, 2, 3, 4, 0, 1, 2 . . . (Notice, no 5).

Clocking a Problem

This new clock face causes new congruence relations. Now we find congruent numbers by adding 5 or a multiple of 5. Thus 0 = (remember everytime you see an equal sign in this article say congruent) 5, 10, 15, 20 . . .; 2 = 7, 12, 17, 22 . . ., etc.

We can perform our operations with these numbers by using this table below or by utilizing the congruence relation.

Modular addition may be defined as "the process of determining the final position of the hand after it has been moved from 0 to the number indicated by the first number in the problem and then advanced the number of hours indicated by the second number in the problem."

THIS SOUNDS COMPLICATED but is really very simple, to add $3 + 4$ in modulo 5, just start at 0, move the hand to 3, then advance it 4 more hours and it will come to rest on 2. Thus $3 + 4 = 2$.

You have a choice of three methods of working this problem: First, you can visualize the movements, or sketch the clock and draw the movement; second you can use the table above; third you can employ congruence relation. $3 + 4$ always equals 7 in ordinary experience, but 7 is congruent to 2 (modulo 5) so write $3 + 4 = 2$.

Some Practical Applications

The same three choices exist in multiplication. To multiply 3×4 in modulo 5, visualize the hand moving from 0 to 3 then being moved ahead 3 hours at a time for a total of 4

CONGRUENCE TABLES MODULO 5

×	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

moves, finally coming to rest at 2. The table shows $3 \times 4 = 2$ directly. The congruence relationship can be used thus: $3 \times 4 = 12$ (Remember to read the = sign as congruent) but 12 is congruent to 2, therefore $3 \times 4 = 2$.

Subtraction is the inverse of addition so it involves counterclockwise movement of the hands, and inverse use of the table. In the example $3 - 4$ the congruence relation can be used conveniently. Because we think of 4 as larger than 3 we can use 8, the next number congruent to 3 and make the problem $8 - 4 = 4$, thus $3 - 4 = 4$ (modulo 5). Of course, in the case of $4 - 3 = 1$ straight arithmetic suffices.

CONGRUENCE IS ALSO convenient in division. In the problem $2 \div 3$ we try successive numbers congruent to 2 until we find one that is divisible by 3. Thus $2 \div 3 = 7 \div 3 = 12 \div 3 = 4$.

In modular arithmetic we can use only integers. Fractions do not exist, so division problems must come out even or not at all.

A return to modulo 12 as used in the early part of this article will bring out some interesting facts about division. Use the congruence relation in the problem that was given earlier, $3 \div 3 = 1$ or 5 or 9. It comes out like this: $3 \div 3 = 1 = 15 \div 3 = 5 = 27 \div 3 = 9 = 39 \div 3 = 13$ but $13 = 1$ so we are beginning to repeat our answers and can stop.

WHEN THE MODULUS is divisible by one of the numbers on the clockface we have a situation where some division problems do not have unique answers. Some have several answers and some have no answer. When the modulo cannot be divided by any number on the clock face the division problems have unique answers.

This modular arithmetic has practical applications. Problems dealing with days of the week are modulo 7. Problems dealing with months of the year are modulo 12 problems.

The big virtue of this topic is that it demonstrates the existence of a totally different kind of number, but one which obeys the communtative, the associative, and distributive laws. The fundamental operations are defined as in regular arithmetic. It is a device to broaden the student's knowledge and appreciation of numbers.

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(TOMORROW: Test Yourselves)

Missionary Unit Plans 2nd Plane

United Missionary Air Training and Transport, with headquarters at the University of Dayton, will put a second plane into service in Africa later this year, according to an announcement from UMATT officials.

The plane, a Cessna 185, will be used to transport supplies and personnel to missionary stations.

John W. St. Andre of East Douglas, Mass., is now completing his training here and will go to Africa to fly the new plane.

UMATT is an interfaith venture that has flown over 150,000 miles last June serving medical workers, missionaries, Peace Corps workers and the people of the newly emerging African nations.

The organization's first aircraft was placed in service last year.